

Kent Maths Puzzle – Christmas Special – Solution

Counting the cost of the twelve days of Christmas

Being a romantic at Christmas time, you want to replicate the gifts given in the carol “The Twelve Days of Christmas” to woo your true love. You have discovered that each group of gifts of the same type cost the same amount, for example, the 12 partridges in pear trees will cost the same as the 22 turtle doves. Given that each individual item has its own price (in pounds and pence) what is the minimum you can expect to pay in total?

Solution:

£3326.40

Explanation from Dr Daniel Bearup:

The trick to this puzzle is to realise that, since the prices of the individual items must cost an exact number of pence, the minimum cost of a group of items must be divisible by the number of items in the group. This allows us to use prime factorisation to solve the problem.

We have been told that each group of gift costs the same amount X and that each group contains a different number of gifts. We can write this down as:

$X = 12 \cdot 1 \cdot y_1 = 11 \cdot 2 \cdot y_2 = 10 \cdot 3 \cdot y_3 = 9 \cdot 4 \cdot y_4 = 8 \cdot 5 \cdot y_5 = 7 \cdot 6 \cdot y_6$, where y_1 is the price for a partridge in a pear tree, or a drummer, y_2 is the price of a turtle dove, or a piper, etc. Thus, the item costs are given by:

Item	Number of Items	Cost of Item	Prime factors
partridge/drummer	12	$X/12$	$2 \cdot 2 \cdot 3$
turtle dove/piper	22	$X/22$	$2 \cdot 11$
French hen/lord	30	$X/30$	$2 \cdot 3 \cdot 5$
calling bird/lady	36	$X/36$	$2 \cdot 2 \cdot 3 \cdot 3$
ring/maid	40	$X/40$	$2 \cdot 2 \cdot 2 \cdot 5$
goose/swan	42	$X/42$	$2 \cdot 2 \cdot 7$

We need to find the smallest X which gives us a whole number when divided by each of these numbers. We could get a suitable X by multiplying each of the group sizes together, but this will end up costing us much more than it needs to. We might note, for a start, that each of the group sizes is divisible by 2, so X is a multiple of 2 but does not need to be a multiple of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$.

To avoid unnecessary duplication of this sort, we can write out the prime factors (see above) of each group size and look for common factors. We see that each group size is a multiple of some combination of the first 5 prime numbers: 2, 3, 5, 7, 11. Additionally, we need three 2's (from gold rings/maids) and two 3's (from calling birds/ladies dancing), and one each of 5, 7, and 11.

The minimum value of X is thus $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. (We can see that each group size will cancel some of these numbers leaving a whole number. If any factor was missing, something would not cancel out and we would be left with a fraction.)

Yet, the question asked was what the total cost is, with 12 days of Christmas and 12 groups of presents. So, the total cost is $12 * X = 2^5 * 3^3 * 5 * 7 * 11 = 332640$. Or, converting to pounds and pence £3326 and 40p. Perhaps it is time to rethink these presents!